Journal of Mathematical Sciences: Advances and Applications
Volume 14, Number 2, 2012, Pages 71-79

MATHEMATICAL INVERSE PROBLEM OF MAGNETIC FIELD FOR 2-DIMENSIONAL EXPONENTIALLY CONDUCTIVITY GROUND PROFILE

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Abstract

Magnetic field response due to the injection of electric current into the ground can be used to explore the earth structure. We derive analytical solutions of the steady state magnetic field due to a direct current source on a continuous conductivity earth structures. A 2-dimensional exponentially varying conductivity of the ground is used in our study. Our solutions in the form of

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2010 Mathematics Subject Classification: 86A25.

Keywords and phrases: integral transformations, inverse problem, magnetic field.

This research is supported by Faculty of Science, Silpakorn University and Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

Received March 15, 2012

magnetic field are obtained by solving a boundary value problem in the wave number domain and then transforming the solution in the wave number domain back to the spatial domain. An inverse problem via the use of the Newton-Raphson optimization technique is introduced for finding the conductivity parameter. The optimal result of our model is close to the true value after using only 6 iterations.

1. Introduction

In geophysical explorations, the traditional resistivity method maps the electrical properties of the earth by measuring the differences in potential caused by a direct current flow between two current electrodes at the earth's surface. Usually, interpretations of electrical soundings are conducted by assuming that the earth's structure consists of horizontally stratified layers having 1-dimensional constant conductivities. A layered earth model is used to simulate the stratigraphic target. However, in the real earth situation, there are cases, where the subsurface conductivity varies continuously rather than discontinuously with depth. This problem was first treated by Mallick and Roy [5], who obtained a theoretical solution for the problem of a 1-dimensional two-layered earth with transitional boundary. Jain [3] presented expressions for a 1-dimensional apparent resistivity of a three-layered earth, where the conductivity in the second layer varies linearly with depth and changes abruptly at the boundaries. Koefoed [4] solved the problem of a 1-dimensional layered earth model containing an arbitrary number of homogeneous layers and of transitional layers in which, the resistivity vary linearly with depth. Banerjee et al. [1] obtained expressions for a 1-dimensional apparent resistivity of a multilayered earth with a layer having binomially varying conductivity.

In this paper, the electrical exploration method based on the measurement of low-level, low-frequency magnetic fields associated with non-inductive current flow between two current electrodes is introduced. Analytical solutions of the steady state magnetic field due to a direct current source on a half space with 2-dimensional exponentially varying conductivities with radial and depth are derived. The method of separation of variables is introduced to our problems and analytical results are obtained with the used of integral transformation. The inversion process, using the Newton-Raphson method, is conducted to estimate a conductivity parameter of the ground.

2. Model and Basic Equations

In our geometric model, a point source of direct current I is located at the interface between two half-spaces. The half-space above the interface $(z < 0)$ is the region of air with conductivity approximately equal to zero, whereas the half-space below the interface is a continuous conductivity earth with depths $(z > 0)$. The conductivity of this half-space is denoted as a function of both radial r and depth z. The general steady state Maxwell's equations in the frequency domain [9] can be used to determine the magnetic field for this problem, namely,

$$\nabla \times \vec{E} = \vec{0}, \tag{1}$$

and

$$\nabla \times \vec{H} = \sigma \vec{E}, \tag{2}$$

where $\vec{E}$ is the vector electric field intensity, $\vec{H}$ is the vector magnetic field intensity, and σ is the conductivity of the medium. Elimination of $\vec{E}$ from Equations (1) and (2), we obtain

$$\nabla \times \frac{1}{\sigma} \nabla \times \vec{H} = \vec{0}. \tag{3}$$

In cylindrical coordinate system (r, ϕ, z), the components in the direction of unit vector $\hat{e}_r, \hat{e}_\phi$, and $\hat{e}_z$ of Equation (3) can be written, respectively, as

$$\frac{1}{r}\frac{\partial}{\partial \phi}\frac{1}{\sigma}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rH_\phi\right) - \frac{1}{r}\frac{\partial H_r}{\partial \phi}\right] - \frac{\partial}{\partial z}\frac{1}{\sigma}\left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right) = 0;$$

and

$$\frac{\partial}{\partial z}\frac{1}{\sigma}\left(\frac{1}{r}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right) - \frac{\partial}{\partial r}\frac{1}{\sigma}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rH_\phi\right) - \frac{1}{r}\frac{\partial H_r}{\partial \phi}\right] = 0; \tag{4}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{1}{\sigma}\left(\frac{\partial H_r}{\partial z}-\frac{\partial H_z}{\partial r}\right)\right]-\frac{1}{r}\frac{\partial}{\partial \phi}\frac{1}{\sigma}\left(\frac{1}{r}\frac{\partial H_z}{\partial \phi}-\frac{\partial H_\phi}{\partial z}\right)=0,$$

where H_r, H_ϕ, and H_z are represented to the components of $\vec{H}$ in the direction of $\hat{e}_r$, $\hat{e}_\phi$, and $\hat{e}_z$, respectively. Since the problem is axi-symmetric and $\vec{H}$ has only the azimuthal component in cylindrical coordinate system for simplicity, we use H to represent the azimuthal component in the following derivations and obtain:

$$-\frac{\partial}{\partial z}\frac{1}{\sigma}\frac{\partial H}{\partial z}-\frac{\partial}{\partial r}\frac{1}{\sigma r}\frac{\partial}{\partial r}(rH)=0,$$

or

$$\frac{1}{\sigma}\frac{\partial^2 H}{\partial z^2}+\frac{\partial}{\partial z}\left(\frac{1}{\sigma}\right)\frac{\partial H}{\partial z}+\frac{1}{\sigma r}\frac{\partial H}{\partial z}-\frac{H}{\sigma^2 r}\frac{\partial \sigma}{\partial r}-\frac{H}{\sigma r^2}+\frac{\partial}{\partial r}\left(\frac{1}{\sigma}\right)\frac{\partial H}{\partial r}+\frac{1}{\sigma}\frac{\partial^2 H}{\partial r^2}=0,$$

where $H = H(r, \phi, z) = H_\phi$.

In our study now, the conductivity of the ground is denoted as a 2-dimensional exponential function of r and z such that $\sigma(r, z) = \sigma_1 e^{b(r+z)}$. Thus, the above partial differential equation becomes

$$\frac{\partial^2 H}{\partial z^2}-b\frac{\partial H}{\partial z}+\frac{1}{r}\left(\frac{\partial H}{\partial r}\right)-\frac{Hb}{r}-\frac{H}{r^2}-\left(b\frac{\partial H}{\partial r}\right)+\frac{\partial^2 H}{\partial r^2}=0. \tag{5}$$

Let the solution of Equation (5) is denoted as

$$H = Z(z)R(r),$$

where $Z(z)$ is a function of z and $R(r)$ is a function of r. Substituting $H = Z(z)R(r)$ into Equation (5) and dividing the result by $Z(z)R(r)$, which we now obtain

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}-\frac{b}{Z}\frac{\partial Z}{\partial z}+\frac{1}{rR}\frac{\partial R}{\partial r}-\frac{b}{r}-\frac{1}{r^2}-\frac{b}{R}\frac{\partial R}{\partial r}+\frac{1}{R}\frac{\partial^2 R}{\partial r^2}=0.$$

The above equation can be separated to

$$\frac{\partial^2 Z}{\partial z^2} - b\frac{\partial Z}{\partial z} - \lambda^2 Z = 0, \tag{6}$$

and

$$\frac{\partial^2 R}{\partial r^2} + \left(\frac{1}{r} - b\right)\frac{\partial R}{\partial r} - \left(\frac{1}{r^2} + \frac{b}{r} + \lambda^2\right)R = 0, \tag{7}$$

where λ is a constant. The solution of (6) is given by

$$Z(z) = A_1 e^{m_1 z} + A_2 e^{m_2 z}, \tag{8}$$

where A_1 and A_2 are arbitrary constants, and $m_1 = \frac{b + \sqrt{b^2 + 4\lambda^2}}{2}$ and $m_2 = \frac{b - \sqrt{b^2 + 4\lambda^2}}{2}$. The method of variation of parameter is introduced to obtain solution of (7) as

$$R(r) = \frac{B_1}{r} + \frac{B_2(br-1)e^{br}}{r} + \frac{\lambda^2}{r}\int_0^r \frac{R(t)(bt-1)}{\left(b^2 t + b - \frac{1}{t}\right)}dt$$

$$+ \frac{\lambda^2(br-1)e^{br}}{r}\int_r^\infty \frac{R(t)}{e^{bt}\left(b^2 t + b - \frac{1}{t}\right)}dt, \tag{9}$$

where B_1 and B_2 are arbitrary constants. The condition of H tends to zero as r and z go to infinity. The solution of (5) is obtained from (8) and (9) via the integral transformation [6, 7] as

$$H(r, z) = \int_0^\infty Z(z, \lambda)R(r, \lambda)d\lambda$$

$$= \int_0^\infty \frac{Ce^{m_2 z}}{r\left(1 - \frac{\lambda^2 \Delta d_{11}}{r}\right)}d\lambda, \tag{10}$$

where C is an arbitrary constant, Δ is the interval of partition in the direction of r, and

$$d_{11} = \lim_{N\to\infty}\sum_{i=1}^{N}\frac{(bt_i - 1)}{\left(b^2 t_i + b - \frac{1}{t_i}\right)}.$$

The arbitrary constant C can be determined from the boundary condition. That is, the current density is zero at the ground surface, except for at the probe source and can be obtained as

$$C = \frac{I\lambda J_0(\lambda r)}{2\pi\left\{\frac{1}{\left(r^2 - \lambda^2\Delta d_{11} r\right)} - \frac{1}{\left(r - \lambda^2\Delta d_{11}\right)^2}\right\}}.$$

3. Numerical Experiment and Inversion Process

In the nature, the variation of conductivity of the ground profile can stand for the weathered zone such that near sea shore areas, where the degree of weathering diminishes with depth. For simplicity, the electrical conductivity is assumed to be exponentially dependent upon the radial r and the depth z, and can be frequently denoted by $\sigma(r, z) = \sigma_1 e^{b(r+z)}$, where σ_1 and b are conductivity parameters of the ground. In our inverse model example, we simulate the reflection of magnetic radiation data from our forward model of the earth structure. Chave's algorithm [2] is used for numerically calculating the integral transform of the magnetic field solution. The special functions are computed by using the numerical recipes source codes (Press et al. [8]). The electric current of 1 ampere is used in our computations. The example model is a heterogeneous conductive half-space having exponentially varying conductivity given by

$$\sigma(r, z) = e^{0.5(r+z)}\, S / m.$$

The conductivity parameter $\sigma_1 = 1 S / m$, which is assumed to be known from the measurement at the probe source on the ground surface. The iterative procedure using the Newton-Raphson method is applied to estimate the model parameter b of conductivity variation. Random errors up to 3% are superimposed on the magnetic field from the forward problem to simulate the set of real data. We start the iterative process to find the value of the conductivity parameter with an initial guess $0.2 S / m.$ The optimal result converges very fast to the true value with percentage error less than 1% after using only 6 iterations as shown in Table 1. The graph of the true and calculated conductivity models are plotted as shown in Figure 1. It is clear that the graph of the calculated model is very close to the true model. This illustrates the advantage in using Newton-Raphson method.

Table 1. Perform the number of iterations used to compute the conductivity parameter b with sum of square error for magnetic field

No. of iteration	1	2	3	4	5	6
Conductivity parameter b	0.200	0.801	0.404	0.599	0.520	0.499
Sum of square error of magnetic field	8.955491	0.022926	0.004391	0.002553	0.000172	0.000011

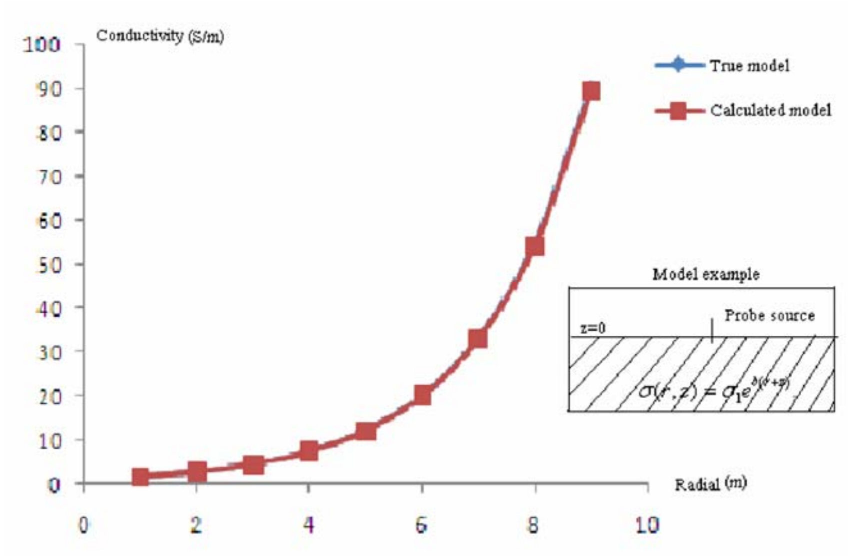

Figure 1. Relation between conductivities and source-receiver spacing of true and calculated models at $z = 0m$.

4. Conclusion

Analytical solutions of the steady state magnetic field due to a direct current source are derived. The 2-dimensional exponentially conductivity model for ground structure is used in our study. The simple method of separation of variables is used to solve partial differential equation with the integral transformations. The inversion process, using the Newton-Raphson method, is conducted to estimate a conductivity parameter of the ground. The optimal result is close to the true value after using only 6 iterations. The method leads to good result and has very high speed of convergence.

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